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APPLICATION OF THE ASYMPTOTIC METHOD OF CERTAIN PROBLEMS OF THE DYNAMICS OF VEHICLES

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APPLICATION OF THE ASYMPTOTIC METHOD OF CERTAIN

PROBLEMS OF THE DYNAMICS OF VEHICLES

V. A. Yaroshevskiy

The present article generalizes and reduces to a compact form the rule of the conservation of the "adiabatic invariant."

As an example of applying the formulas obtained, we will examine problems concerning orbital decay of a satellite, the skip-glide motion of a glider, and the oscillations of a vehicle about the center of mass on a flight path with a variable velocity.

1. When investigating a number of questions concerning the dynamics of vehicles, the problem reduces to finding a solution of a system of equations in which the principal second-order equation describes a rapid quasiperiodic motion, and the parameters of the system are variable, slowly changing magnitudes whose change is described by a system of second-order equations.

Let us examine the system of equations

$$\frac{d^2y}{dt^2} + F\left(\tau,\ r_j,\ y,\ \varepsilon\frac{dy}{dt}\right) = 0, \tag{1}$$

$$\frac{dr_i}{dt} - \varepsilon s_i\left(\tau,\ r_j,\ y,\ \frac{dy}{dt}\right) = 0 \quad i = 1, 2, \ldots n. \tag{2}$$

Here ε is the parameter of smallness, $\tau = \varepsilon t$ is "slow" time, F and s_1 are differentiable functions of both arguments.

At constant values of τ and r_j and when $\varepsilon = 0$, the first equation describes a rapid periodic motion.

To obtain the relationships characterizing the change of "amplitude" values of the variable y and the "mean" values of the variables r_1, we will use in this work a method analogous to the method developed by Kuzmak [1, 2].

We will represent the unknown functions as functions of two variables, "slow" time τ and the phase of the periodic motion φ:

$$y = y_0(\tau, \varphi) + \varepsilon y_1(\tau, \varphi) + 0(\varepsilon^2), \qquad r_i = r_{i0}(\tau) + \varepsilon r_{i1}(\tau, \varphi) + 0(\varepsilon^2),$$
$$\frac{d\varphi}{dt} = \omega_0(\tau) + \varepsilon\omega_1(\tau) + 0(\varepsilon^2).$$

To avoid the appearance of secular members, we will require that the functions be periodic with respect to φ. Then the order of smallness of the terms of the asymptotic expansion will be retained in the interval $\tau \sim 1$, i.e., $t \sim 1/\varepsilon$ [1].

Separating in all equations the terms of the zero and first order of smallness, we derive

$$\begin{aligned} &\omega_0^2 \frac{\partial^2 y_0}{\partial\varphi^2} + F[\tau, r_{j0}, y_0, 0] + \varepsilon \left\{\frac{d\omega_0}{d\tau}\frac{\partial y_0}{\partial\varphi} + 2\omega_0 \frac{\partial^2 y_0}{\partial\varphi\,\partial\tau} + \frac{\partial F}{\partial y'}(\tau, r_{j0}, y_0, 0)\,\omega_0 \frac{\partial y_0}{\partial\varphi} + \right. \\ &\quad + \omega_0^2 \frac{\partial^2 y_1}{\partial\varphi^2} + \frac{\partial F}{\partial y}(\tau, r_{j0}, y_0, 0)\,y_1 + 2\omega_0\omega_1 \frac{\partial^2 y_0}{\partial\varphi^2} + \\ &\quad \left. + \sum_{j=1}^{n} \frac{\partial F}{\partial r_j}(\tau, r_{j0}, y_0, 0)\,r_{j1}\right\} + 0(\varepsilon^2) = 0. \end{aligned} \tag{3}$$

Here $\partial F/\partial y'$ designates a partial derivative of the function F with respect to the argument $\varepsilon\, dy/dt$.

$$\varepsilon\left\{\omega_0\frac{\partial r_{i1}}{\partial\varphi}+\frac{dr_{i0}}{d\tau}-s_i\left(\tau,\ r_{j0},\ y_0,\ \omega_0\frac{\partial y_0}{\partial\varphi}\right)\right\}+0(\varepsilon^2)=0 \tag{4}$$

Hereafter such partial derivatives of the function F will be designated as $\partial F_0/\partial y$, $\partial F_0/\partial y'$ and $\partial F_0/\partial r_j$, and the functions themselves $F(\tau,\ r_{j0},\ y_0,\ 0)$ and $s_i(\tau,\ r_{j0},\ y_0,\ \omega_0\ \partial y_0/\partial\varphi)$ as F_0 and S_{i0}.

Integrating the expression in ε in Eq. (4) with respect to φ from φ to $\varphi + 2\pi$ and taking into account that r_{i1} is a function periodic with respect to φ, we obtain

$$\frac{dr_{i0}}{d\tau}-\overline{s_{i0}}=0, \tag{5}$$

where

$$\overline{s_{i0}}=\frac{1}{2\pi}\int\limits_{\varphi}^{\varphi+2\pi}s_i\left(\tau,\ r_{j0},\ y_0,\ \omega_0\frac{\partial y_0}{\partial\varphi}\right)d\varphi$$

(naturally, when integrating here with respect to φ the quantities τ and r_j are considered parameters).

The term $r_{j1}(\tau,\ \varphi)$ is determined by the relation:

$$r_{i1}(\tau,\ \varphi)=\frac{1}{\omega_0}\int\limits_{\varphi_0(\tau)}^{\varphi}[s_{i0}-\overline{s_{i0}}]\,d\varphi, \tag{6}$$

where $\varphi_0(\tau)$ is some function which is nonessential when calculating the zero terms of the asymtotic expansion.

The term y_0 is a solution of the "standard" equation

$$\omega_0^2(\tau)\frac{\partial^2 y_0}{\partial\varphi^2}+F[\tau,\ r_{j0},\ y_0,\ 0]=0. \tag{7}$$

The value of ω_0 is conveniently determined from the hypothesis that $y_0(\varphi) = y_0(\varphi + 2\pi)$.

The change in time of the amplitude values y_0, which is of particular interest to us, cannot be determined from this equation, therefore we must attract the "condition of periodicity" of the function y_1 [1].

The equation (linear) for determining the function y_1 has the following form (see Eq. (3)):

$$\omega_0^2(\tau)\frac{\partial^2 y_1}{\partial\varphi^2}+\frac{\partial F_0}{\partial y}y_1=$$
$$=-\left[\frac{d\omega_0}{d\tau}\frac{dy_0}{d\varphi}+2\omega_0\frac{\partial^2 y_0}{\partial\varphi\,\partial\tau}+\frac{\partial F_0}{\partial y'}\omega_0\frac{\partial y_0}{\partial\varphi}+2\omega_0\omega_1\frac{\partial^2 y_0}{\partial\varphi^2}+\sum_{j=1}^{n}\frac{\partial F_0}{\partial r_j}r_{j1}\right] \tag{8}$$

Differentiating Eq. (7) in terms of φ, we are easily convinced that the term $\partial y_0/\partial\varphi$ satisfies Eq. (8) without the right part. Then, multiplying Eq. (7) by y_1 and (12) by $\partial y_0/\partial\varphi$, subtracting the obtained relations from one another, integrating in terms of φ from φ to φ + 2π and taking into account that $\int\limits_{\varphi}^{\varphi+2\pi}\frac{\partial^2 y_0}{\partial\varphi^2}\frac{\partial y_0}{\partial\varphi}d\varphi=0$, we easily find the first conditions needed for the periodicity of the function y_1 in terms of φ:

$$\int\limits_{\varphi}^{\varphi+2\pi}\left\{\frac{d\omega_0}{d\tau}\frac{\partial y_0}{\partial\varphi}+2\omega_0\frac{\partial^2 y_0}{\partial\varphi\,\partial\tau}+\frac{\partial F_0}{\partial y'}\omega_0\frac{\partial y_0}{\partial\varphi}+\right.$$
$$\left.+\sum_{j=1}^{n}\frac{\partial F_0}{\partial r_j}r_{j1}\right\}\frac{\partial y_0}{\partial\varphi}d\varphi=0 \tag{9}$$

We will not examine the second condition of periodicity which enables us to determine the correction ω_1.

Taking into account that y_0 is a solution of Eq. (7), we find that

$$\frac{\partial y_0}{\partial\varphi}=\pm\frac{1}{\omega_0}\left(-2\int\limits_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}, \tag{10}$$

where y_m is the amplitude value of the function y_0 indifferently, more or less, corresponding to the vanishing of the derivative $\partial y_0/\partial\varphi$ (the plus sign is ascribed to the increasing portion of y_0 and the minus sign to the decreasing portion of y_0).

It is evident that relation (10) is equivalent to the determination of the "instantaneous amplitude y_m for values y and dy/dt by means of "frozen" Eq. (1) (τ = const, r_j = const, $\varepsilon = 0$).

We will transform Eq. (9) to the following form:

$$\frac{d}{d\tau}\left[\int_{\varphi}^{\varphi+2\pi}\omega_0\left(\frac{\partial y_0}{\partial\varphi}\right)^2 d\varphi\right]+\int_{\varphi}^{\varphi+2\pi}\omega_0\frac{\partial F_0}{\partial y'}\left(\frac{\partial y_0}{\partial\varphi}\right)^2 d\varphi+$$
$$+\sum_{l=1}^{n}\int_{\varphi}^{\varphi+2\pi}\frac{\partial F_0}{\partial r_l}\frac{\partial y_0}{\partial\varphi}r_{l1}\,d\varphi=0.$$

When integrating with respect to φ, the magnitudes τ and r_{j0} must be considered as parameters, therefore $\partial y_0/\partial\varphi d\varphi = dy_0$. Taking this into account as well as relation (10), we will reduce Eq. (9) to the following form:

$$\frac{d}{d\tau}\oint\left(-2\int_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}dy_0+$$
$$+\oint\frac{\partial F_0}{\partial y'}\left(-2\int_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}dy_0+\sum_{l=1}^{n}\oint\frac{\partial F_0}{\partial r_l}r_{l1}\,dy_0=0. \qquad (11)$$

(The sign $\oint$ means that integration is performed for the complete period of the change y_0).

Integrating by parts and taking into account that r_{j1} is a periodic function determined by formula (6), and

$$\oint\frac{\partial F_0}{\partial r_l}dy_0=0,$$

we obtain

$$\oint\frac{\partial F_0}{\partial r_l}r_{l1}\,dy_0=\oint\left(-\int_{y_m}^{y_0}\frac{\partial F_0}{\partial r_l}dy_0\right)\frac{s_{l0}-\bar{s}_{l0}}{\left(-2\int_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}}dy_0. \qquad (12)$$

Having differentiated the first term in Eq. (11), we derive

$$\frac{d}{d\tau}\oint\left(-2\int_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}dy_0=\oint\frac{\sum\limits_{j=1}^{n}\left(-\int\limits_{y_m}^{y_0}\frac{\partial F_0}{\partial r_j}dy_0\right)\bar{s}_{j0}}{\left(-2\int\limits_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}}dy_0+$$

$$+\frac{\partial}{\partial\tau}\bigg|_{r_{j0}=\text{const}}\left[\oint\left(-2\int_{y_m}^{y_0}F_0\,dy_0\right)^{1/2}dy_0\right]. \qquad (13)$$

Substituting (12) and (13) into (11), we obtain

$$\frac{\partial}{\partial\tau}\bigg|_{r_{j0}=\text{const}}\oint\frac{dy_0}{dt}(y_0)\,dy_0+\oint\frac{\partial F_0}{\partial y'}\frac{dy_0}{dt}(y_0)\,dy_0+$$

$$+\oint\frac{\sum\limits_{j=1}^{n}\left(-\int\limits_{y_m}^{y_0}\frac{\partial F_0}{\partial r_j}dy_0\right)s_{j0}\,dy_0}{\frac{dy_0}{dt}(y_0)}=0. \qquad (14)$$

Here $dy_0/dt(y_0)$ is the solution of Eq. (1) with frozen values of r_{j0} and τ when $\varepsilon = 0$.

Since

$$\int_{y_m}^{y_0}\frac{\partial F_0}{\partial r_j}dy_0=-\frac{1}{2}\frac{\partial}{\partial r_j}\left[\frac{dy_0}{dt}(y_0)\right]^2=-\frac{dy_0}{dt}\frac{\partial}{\partial r_j}\left[\frac{dy_0}{dt}\right],$$

then the third integral in (14) can be presented in the following form:

$$\oint\frac{\sum\limits_{j=1}^{n}\left(-\int\limits_{y_m}^{y_0}\frac{\partial F_0}{\partial r_j}dy_0\right)s_{j0}\,dy_0}{\frac{dy_0}{dt}(y_0)}=\oint\sum_{j=1}^{n}\frac{\partial}{\partial r_j}\left[\frac{dy_0}{dt}(y_0)\right]s_{j0}\,dy_0.$$

We will introduce the following designations for averageable values:

$$\overline{\overline{\frac{\partial F_0}{\partial y'}}} = \frac{\oint \frac{\partial F_0}{\partial y'} \frac{dy_0}{dt} dy_0}{\oint \frac{dy_0}{dt} dy_0} = \overline{\overline{\frac{\partial F_0}{\partial y'}}}(\tau, r_{j0}, y_m),$$

$$\overline{\overline{s_{j0}}} = \frac{\oint \frac{\partial}{\partial r_j}\left[\frac{dy_0}{dt}\right] s_{j0}\, dy_0}{\oint \frac{\partial}{\partial r_j}\left[\frac{dy_0}{dt}\right] dy_0} = \overline{\overline{s_{j0}}}(\tau, r_{j0}, y_m),$$

$$\oint \frac{dy_0}{dt} dy_0 = D(\tau, r_{j0}, y_m)$$

is the "adiabatic invariant" [3].

Then the concluding result can be presented in the following form:

$$\frac{\partial}{\partial \tau}\bigg|_{r_{j0}=\mathrm{const}} D + \sum_{j=1}^{n} \frac{\partial D}{\partial r_{j0}} \overline{\overline{s_{j0}}} + D \overline{\overline{\frac{\partial F_0}{\partial y'}}} = 0. \qquad (15)$$

The values of r_{jo} are determined from Eq. (5)

$$\frac{dr_{j0}}{d\tau} = \bar{s}_{j0}(\tau, r_{j0}, y_m).$$

Using these equations we can determine the change in time of the amplitude of the zero term of the asymptotic expansion for y (i.e., y_m) and also the change in time of the zero terms of the asymptotic expansion (mean values) for r_1 (i.e., r_{1o}). The difference between the exact solution and the zero terms of the asymptotic expansion in the interval $t \sim 1/\varepsilon$ retains an order of smallness ε.

The first two terms of Eq. (15) can be combined into a "unique derivative" of the integral D in time, which is distinguished from the real, complete derivative in that the values $dr_{jo}/dt = \overline{s_{jo}}$ are replaced by the values $\overline{\overline{s_{jo}}}$. If the right-hand parts of Eq. (2) were to depend on y and dy/dt, then (when $\partial F/\partial y' = 0$) we would obtain the known rule of the conservation of the adiabatic invariant D = const [3]. Equation (15) generalizes this rule.

The magnitude $\overline{\overline{\partial F_0/\partial y'}}$ corresponds to the term $\partial F_0/\partial y'$ averaged

over a period with a weight of $(dy_0/dt)^2$, the magnitudes $\overline{\overline{s_{j0}}}$ correspond to the terms s_{j0} averaged over a period with a weight

$$\frac{\partial}{\partial r_j}\left[\left(\frac{dy_0}{dt}\right)^2\right].$$

It is easy to see that in the important special case where

$$F\left(\tau,\, r_j,\, y,\, \varepsilon\,\frac{dy}{dt}\right) = F_1\left(\tau,\, y,\, \varepsilon\,\frac{dy}{dt}\right) F_2(\tau, r_j),$$

the magnitude $\overline{\overline{s_{j0}}}$, like $\overline{\overline{\partial F_0/\partial y'}}$, is a magnitude averaged over a period with a weight of $(dy_0/dt)^2$.

We note in conclusion that these results can easily be generalized for the case where the term $\varepsilon\, dy/dt$ is replaced by the term $\varepsilon f(dy/dt)$.

2. Let us examine, as the first example, the problem of the decay of an elliptical orbit of a satellite owing to a small aerodynamic drag.

The initial system of equations is written in the following form [4]:

$$\frac{d^2\xi}{d\varphi^2} + \xi = \eta, \tag{16}$$

$$\frac{d\eta}{d\varphi} = \eta\,\frac{C_x S R_0}{m}\,\frac{\rho(\xi)}{\xi}\sqrt{1 + \left(\frac{1}{\xi}\,\frac{d\xi}{d\varphi}\right)^2}, \tag{17}$$

where

$$\xi = \frac{R_0}{R}, \quad \eta = \frac{g_0 R_0^3}{\left(R^2\,\frac{d\varphi}{dt}\right)^2}$$

is the magnitude which is the invariant on a Keplerian trajectory and in this case slowly changes, R is the distance from the vehicle to the center of the planet, R_0 is the planet radius, g_0 is the acceleration of gravity at the planet surface, φ is the central angle, ρ is the density of the atmosphere, C_x is the drag coefficient, S is the

characteristic area, m is the satellite mass. The meaning of the designations can be cleared up from Fig. 1. The perigean point corresponds to the value $\xi_p = \frac{R_0}{R_p}$.

Let us apply the method cited in part 1 to this system (16), (17). Taking into account that

$$\frac{d\xi}{d\varphi}(\xi_p, \eta, \xi) = \sqrt{(\xi_p - \eta)^2 - (\xi - \eta)^2}, \tag{18}$$

we will find the value of the adiabatic invariant

$$D = \oint \frac{d\xi}{d\varphi} d\xi = \pi \quad - \eta)^2:$$

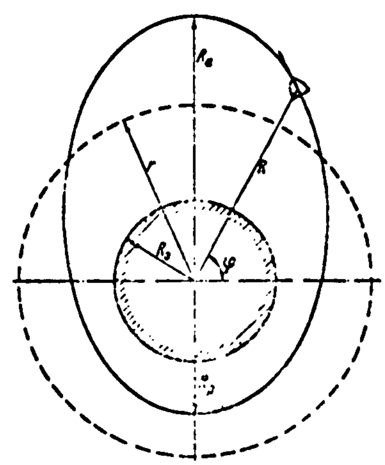

Fig. 1.

Equation (15) has the following form here:

$$\frac{\partial D}{\partial \xi_p}\frac{d\xi_p}{d\varphi} + \frac{\partial D}{\partial \eta}\frac{\overline{\overline{\partial \eta}}}{\partial \varphi} = 0.$$

Hence

$$\frac{d\xi_p}{d\varphi} = \frac{\overline{d\eta}}{d\varphi}(\eta_0, \xi_p)$$

(η_0 is the zero term of the asymptotic expansion for η or the average value for η during the circling time).

Equation (5) is written as

$$\frac{d\eta_0}{d\varphi} = \overline{\frac{d\eta}{d\varphi}}(\eta_0, \xi_p).$$

Here

$$\overline{\frac{d\eta}{d\varphi}} = \frac{C_x S R_0}{m} \eta_0 \frac{\int\limits_{2\eta_0 - \xi_p}^{\xi_p} \frac{\rho(\xi)}{\xi} \sqrt{1 + \left(\frac{1}{\xi}\frac{d\xi}{d\varphi}\right)^2} \frac{d\xi}{d\xi | d\varphi}}{\int\limits_{2\eta_0 - \xi_0}^{\xi_p} \frac{d\xi}{d\xi | d\varphi}}, \tag{19}$$

$$\overline{\overline{\frac{d\eta}{d\varphi}}} = \frac{C_x S R_0}{m} \eta_0 \frac{\int\limits_{2\eta_0 - \xi_p}^{\xi_p} \frac{\rho(\xi)}{\xi} \sqrt{1 + \left(\frac{1}{\xi}\frac{d\xi}{d\varphi}\right)^2} \frac{\partial}{\partial \overline{\eta}}\left(\frac{d\xi}{d\varphi}\right) d\xi}{\int\limits_{2\eta_0 - \xi_p}^{\xi_p} \frac{\partial}{\partial \overline{\eta}}\left(\frac{d\xi}{d\varphi}\right) d\xi}. \tag{20}$$

where $d\xi | d\varphi$ are determined by formula (18).

Let us extract the explicit expressions for these derivatives in two extreme cases.

a) Considering the eccentricity of the orbit as small ($\xi_p - \xi \leq \leq \xi_p$), approximating the dependence of density on height $H = R - R_0$ by the function $\rho = \rho_0 \exp(-\lambda H)$, and introducing the concept of an "average" radius of the orbit $r = \frac{R_0}{\eta_0}$, we will transform Eqs. (19) and (20) to the following (dimensional) form:

$$\frac{dR_p}{d\varphi} = -\frac{\rho_0 C_x S r^2}{m} e^{-\lambda(r - R_0)} \{I_0[\lambda(r - R_p)] - I_1[\lambda(r - R_p)]\} \times$$
$$\times \left\{1 + 0\left[\lambda \frac{(r - R_p)^2}{r}\right] + 0\left[\frac{r - R_p}{r}\right]\right\}$$
$$\frac{dr}{d\varphi} = -\frac{\rho_0 C_x S r^2}{m} e^{-\lambda(r - R_0)} I_0[\lambda(r - R_p)] \left\{1 + 0\left[\lambda \frac{(r - R_p)^2}{r}\right] + 0\left[\frac{r - R_p}{r}\right]\right\}.$$

Designating the parameter $\lambda(r - R_p)$, which in the indicated assumption is proportional to the eccentricity, in terms of $\underline{x}$, we derive the known result (see, for example, [5])

$$\frac{dr}{dx} = -\frac{1}{\lambda}\frac{I_0(x)}{I_1(x)},$$

where I_0 and I_1 are Bessel's functions of the imaginary argument.

At large values of x, $I_0|I_1 \to 1$. Consequently, if the eccentricity is comparatively large, the height of the perigee R_p decreases slowly in comparison with the average radius of the orbit r (and the height of the apogee R_a), and the eccentricity rapidly decreases.

At small values of x, $I_0|I_1 \approx 2|x$, $x \sim \exp \lambda r|2$, the orbit changes to a circular orbit, the average radius of the orbit and the height of the perigee virtually coincide and they decrease with the same velocity.

b) If the eccentricity of the original orbit is large, then the main deceleration of the satellite occurs in the portion of the path adjacent to the perigean point, wherein the role of this portion of the path grows with an increase of λ. The dependence of density on height for a large range of altitudes, characteristic for an orbit with a large eccentricity, can be approximated by the function

$$\rho = \rho_0 e^{-\int_0^H \lambda(\varepsilon H)\, dH},$$

where λ is a slowly changing function $\lambda(\varepsilon H)$. In the section $R - R_p \ll \ll R_p$, which is of interest to us, we can consider that

$$\rho = \rho_p e^{-\lambda_p (R-R_p)}$$
$$(\rho_p = \rho(R_p),\ \lambda_p = \lambda(R_p)).$$

Simplifying the expressions for derivatives (19) and (20) in accord with tho assumptions made, we obtain (again in a dimensional form)

$$\frac{dr}{d\varphi} = -\frac{C_x S r \rho_p}{m\sqrt{2\pi\lambda_p\left(\frac{1}{R_p}-\frac{1}{r}\right)}} \times$$

$$\times\left\{1+0\left[\frac{1}{\lambda_p^2}\frac{d\lambda_p}{dH}\right]+0\left[\frac{1}{\lambda_p(R_a-R_p)}\right]+0\left[\frac{R_a-R_p}{\lambda_p R_a R_p}\right]\right\},$$

$$\frac{dR_p}{d\varphi} = -\frac{C_x S \rho_p}{m\,2\sqrt{2\pi}\left[\lambda_p\left(\frac{1}{R_p}-\frac{1}{r}\right)\right]^{3/2}} \times$$

$$\times\left\{1+0\left[\frac{1}{\lambda_p^2}\frac{d\lambda_p}{dH}\right]+0\left[\frac{1}{\lambda_p(R_a-R_p)}\right]+0\left[\frac{R_a-R_p}{\lambda_p R_a R_p}\right]\right\}.$$

A similar result is obtained by a different method in El'yasberg's study [6]. It follows from the reduced relations that

$$\frac{d(1/r)}{dR_p} = -2\lambda(R_p)\left(\frac{1}{R_p}-\frac{1}{r}\right),$$

i.e., we obtain an ordinary linear equation which is easily integrated.

For convenience we can combine the results obtained for small and large eccentricities and write them in the following form:

$$\frac{dr}{d\varphi} = -\frac{C_x S r^2}{m}\rho(R_p) I_0(x) e^{-x}, \qquad \frac{dR_p}{d\varphi} = -\frac{C_x S r^2}{m}\rho(R_p)[I_0(x) - I_1(x)] e^{-x},$$

where $x = \lambda(R_p)\frac{r}{R_p}(r - R_p)$ (at large values of x $I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$, $I_0(x) - I_1(x) \approx \frac{e^x}{\sqrt{\pi}(2x)^{3/2}}$). Since λr has a order of 100-900, we can expect that the ranges of variability of the formulas overlap.

3. As the second example we will examine the motion of a flying vehicle with a large positive quality (the so-called skip trajectory) [7].

We will introduce the following assumptions:

$$H \ll R_0,\ |\theta| \ll 1,\ \rho = \rho_0 e^{-\lambda H},\ C_x = \text{const}.$$

Then the equation of motion can be rewritten in the following form:

$$\frac{d^2H}{ds^2} = \frac{C_y \rho S V^2}{2m} - g_0 + \frac{V^2}{R_0},$$
$$\frac{d(V^2)}{ds} = -\frac{C_x \rho S V^2}{m} - 2g_0 \frac{dH}{ds}.$$

Here H — height, V — velocity, θ — the local angle of inclination of the trajectory, s — a variable of the path, C_x and C_y — coefficients of drag and lift (see, for example, [7]).

Considering that the velocity changes negligibly during the oscillation period with respect to H and that C_y changes slowly, we will apply formulas (5) and (15) to this system of equations.

We are easily satisfied that when calculating the averaged derivatives $\overline{dV^2|ds}$ and $\overline{\overline{dV^2|ds}}$, the term $2gdH|ds$ does not have any effect; it vanished on averaging. Therefore, this term is not taken into account when calculating the amplitude and mean value of the zero terms of the asymptotic expansion for height and velocity respectively.

Then, if we eliminate the variable ds, we derive in plane (H, V) the following equation of motion:

$$\frac{d^2y}{dx^2} = -\sqrt{R_0\lambda}\frac{C_y}{C_x} + \frac{e^{2x}-1}{y},$$

where $x = \ln \frac{\sqrt{R_0 g_0}}{V}$, $y = \frac{C_x S}{2m} \sqrt{\frac{R_0}{\lambda}} \rho$.

We can apply to this equation the method cited in Kuzmak's study [1], considering exp(2x)-1 a slowly changing coefficient, which is valid for large values of the quality and at velocities smaller than circular velocity.

Without going into detailed calculations, we can extract the final formula which determines the dependence of the amplitude values of

height on the mean velocity:

$$\left(\frac{\frac{R_0 g_0}{V_{initial}^2} - 1}{\frac{R_0 g_0}{V^2} - 1}\right)^{3/2} \frac{C_y}{C_{y\,initial}} h\left\{\frac{C_y S R_0 \rho_{max}}{2m\left(\frac{R_0 g_0}{V^2} - 1\right)}\right\} = \text{const.}$$

Here the argument of the function h is the ratio of the maximal amplitude of density ρ_{max} to the density corresponding to a trajectory of quasistationary gliding

$$\rho_* = \frac{2m(R_0 g_0 - V^2)}{C_y S R_0 V^2}.$$

The function h is determined by the formula

$$h(u) = \int_{z_1(u)}^{z_2(u)} \sqrt{u - z + \ln\frac{z}{u}}\, dz.$$

where $z_1(u)$ and $z_2(u) = u$ are the roots of the equation $u - z + \ln\frac{z}{u} = 0$.

The graph of the function h(u) is shown in Fig. 2. When $u \approx 1$ $h(u) \approx \frac{\pi}{2\sqrt{2}}(u - 1)^2$, when $u \gg 1$ $h(u) \approx \frac{2u\sqrt{u}}{3}$.

Hence it is clear that in "deep" skips ($u \gg 1$) when C_y = const, the value of ρ_{max} is practically constant (analogous to the case where the height of the perigee is at first almost constant upon decay of an elliptic orbit with a large eccentricity), which coincides with the results obtained earlier [7]. As the velocity decreases $h \to 0$, i.e., $\rho_{max} \to \rho_*$, the trajectory becomes a trajectory of quasistationary gliding. Here the minimal (amplitudinal) density ρ_{min} is associated with the maximal relation

$$\frac{\rho_{max}}{\rho_*}\exp\left(-\frac{\rho_{max}}{\rho_*}\right) = \frac{\rho_{min}}{\rho_*}\exp\left(-\frac{\rho_{min}}{\rho_*}\right).$$

The instantaneous oscillation "period" with respect to the velocity is determined by the formula

$$T_V = \frac{2\pi C_x}{\sqrt{R_0\lambda}\, C_y} \sqrt{R_0 g_0 - V^2}\, T\left(\frac{\rho_{max}}{\rho_0}\right).$$

The function $T(u) = \frac{\sqrt{2}}{\pi} \frac{u}{u-1} h'(u)$ is shown in Fig. 2. Since $T \geq 1$, the number of complete oscillations of the vehicle when C_y = const does not exceed

$$n = \int_{V_{end}}^{V_{initial}} \frac{dV}{T_V} < \frac{\sqrt{R_0\lambda}\, C_y}{2\pi C_x} \times \arc\sin \frac{V}{\sqrt{R_0 g_0}} \Bigg|_{V_{end}}^{V_{initial}}$$

or

$$n < \frac{\sqrt{R_0\lambda}}{4} \frac{C_y}{C_x} \approx 7{,}5 \frac{C_y}{C_x}.$$

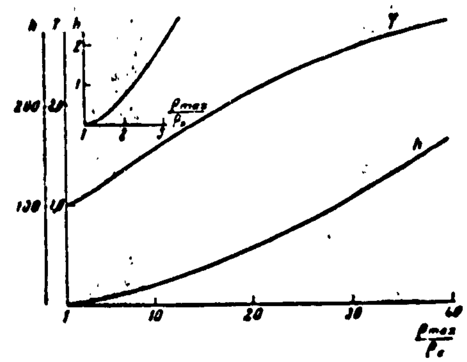

Fig. 2.

4. In the last example we will examine a rapid plane oscillatory motion of a ballistic vehicle in an idealized medium with a constant density in the absence of a gravitational field. For simplicity we will exclude from the examination the damping terms in the equation of the oscillations of the vehicle about the center of mass (although this cannot be done in practical calculations).

The equations are written in the following form:

$$\frac{d^2\alpha}{dt^2} = k_1 m_z^\alpha \alpha V^2,$$

$$\frac{dV}{dt} = -k_2 C_x\ \alpha)\, V^2.$$

Here $k_1 = \frac{\rho S l}{2J}$, $k_2 = \frac{\rho S}{2m}$, α is the angle of attack, m_z^{α} is a derivative of the aerodynamic moment coefficient, C_x is the coefficient of drag, S and l are the characteristic surface and length, J is the moment of inertia, m is mass of the vehicle.

If the coefficient C_x did not depend on α, the change in amplitude of oscillations α_m would be determined by the asymptotic formula

$$\alpha_m^2 \sim \frac{1}{\sqrt{-k_1 m_z^{\alpha} V^2}}$$

or

$$\alpha_m^2 V = \text{const.} \tag{21}$$

We will apply formulas (5) and (15) to the extracted equations. Taking into account

$$D = \sqrt{-k_1 m_z^{\alpha}} V \pi \alpha_m^2,$$

we obtain equations describing the changes of the zero terms of the asymptotic expansion for velocity V_0 and for the amplitude α_m

$$\frac{dV_0}{dt} = -k_2 V_0^2 \frac{\int\limits_{-\alpha_m}^{\alpha_m} \frac{C_x(\alpha)\, d\alpha}{\sqrt{\alpha_m^2 - \alpha^2}}}{\int\limits_{-\alpha_m}^{\alpha_m} \frac{d\alpha}{\sqrt{\alpha_m^2 - \alpha^2}}} = -k_2 V_0^2 \overline{C_x(\alpha_m)},$$

$$V_0 \frac{d}{dt}(\alpha_m^2) + \alpha_m^2 \frac{\overline{dV}}{dt} = 0,$$

$$\frac{\overline{\overline{dV}}}{dt} = -k_2 V_0^2 \frac{\int\limits_{-\alpha_m}^{\alpha_m} C_x(\alpha) \sqrt{\alpha_m^2 - \alpha^2}\, d\alpha}{\int\limits_{-\alpha_m}^{\alpha_m} \sqrt{\alpha_m^2 - \alpha^2}\, d\alpha} = -k_2 V_0^2 \overline{\overline{C_x(\alpha_m)}}.$$

Then the equation relating the amplitude to the mean velocity takes the following form:

$$2d\ln\alpha_m + \frac{\overline{\overline{C_x(\alpha_m)}}}{\overline{C_x(\alpha_m)}}\, d\ln V_0 = 0. \tag{22}$$

If $\overline{\overline{C_x(\alpha_m)}} \approx \overline{C_x(\alpha_m)}$, we derive formula (21). We will approximate the dependence $C_x(\alpha)$ by a function of the form:

$$C_x = a + b|\alpha|^m \quad (a>0,\ b>0).$$

Then

$$\overline{C_x(\alpha_m)} = a + \frac{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\pi\Gamma\left(\frac{m}{2}+1\right)}\, b\alpha_m^m,$$

$$\overline{\overline{C_x(\alpha_m)}} = a + \frac{2\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\pi\Gamma\left(\frac{m}{2}+2\right)}\, b\alpha_m^m.$$

Using these formulas, we can easily integrate Eq. (22) and obtain the relation between V_0 and α_m in an explicit form.

We will limit ourselves to a qualitative investigation of this equation.

When $m > 0$

$$1 > \frac{\overline{\overline{C_x(\alpha_m)}}}{\overline{C_x(\alpha_m)}} > \frac{2}{2+m}.$$

In the limiting case when $a = 0$ and $m \to \infty$ we derive that this relation approaches zero, i.e., $\alpha_m = \text{const}$, in spite of the fact that the velocity decreases monotonically and the relative change of velocity during the oscillation period with respect to α can be as small as desired. The physical sense of the last result is that the fall-off of velocity V mainly occurs only in the neighborhood of amplitude $\alpha = \alpha_m$, and outside this neighborhood the velocity can be considered as a virtually constant value. Therefore the motion "inside" each

half-period is conservative, and the frequency change jumplike from half-period to half-period.

An opposite "degenerate" case can be obtained if we assume that $0 > m > -1$ so that $C_x(\alpha = 0) = \infty$.

Then the relation under consideration is greater than unity, i.e., in this case when $m \rightarrow -1$, the amplitude can increase proportionally $1/V$ (see [22]). Both degenerate cases are illustrated in Fig. 3.

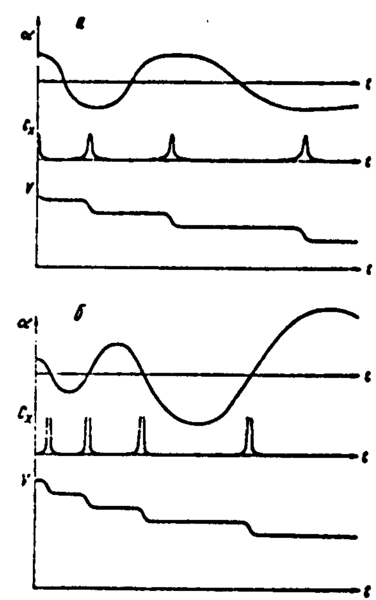

Fig. 3.

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